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Solution by **M. A. GRUBER, A. M.**, War Department, Washington, D. C.; **CHAS. C. CROSS**, Memphis, Tenn.; and **J. E. SANDERS**, Hackney, Ohio.

Let $x-1$, x, and $x+1$=the sides of a rational triangle.

$$\text{Then, } (Area)^2=\frac{3x}{2}\cdot\frac{x}{2}\cdot\frac{x+2}{2}\cdot\frac{x-2}{2}=\frac{3x^2(x^2-4)}{16}=\square.$$

$$\therefore\ 3(x^2-4)=\square=9m^2\text{; and } x=\surd(3m^2+4).$$

To rationalize $\surd(3m^2+4)$, we use the convergents of the $\surd3=\frac{1}{1}$, $\frac{2}{1}$, $\frac{5}{3}$, $\frac{7}{4}$, $\frac{19}{11}$, $\frac{26}{15}$, $\frac{71}{41}$, $\frac{97}{56}$, $\frac{265}{153}$, $\frac{362}{209}$, $\frac{989}{571}$, $\frac{1351}{780}$, etc. Beginning with $\frac{2}{1}$ and taking the alternate convergents, we put x=twice the numerators and m=twice the denominators, and obtain the following set of pairs of values:

x=4, 14, 52, 194, 724, 2702, etc.
m=2, 8, 30, 112, 418, 1560, etc.

$\therefore$ The sides of the first six triangles are 3, 4, 5; 13, 14, 15; 51, 52, 53; 193, 194, 195; 723, 724, 725; 2701, 2702, 2703. The area=$3mx/4$.

Taking x_{n-2} and x_{n-1} as any two consecutive middle sides, we find the next middle side, or $x_n=4x_{n-1}-x_{n-2}$. This law of formation gives the following general expression for the middle side of the nth triangle, using 4 and 2 (=middle side of "straight line" triangle) as the middle sides of two consecutive triangles:

$$x_n=4[4^{n-1}-(n-2)4^{n-3}+\frac{(n-3)(n-4)}{2}4^{n-5}-\frac{(n-4)(n-5)(n-6)}{2\times3}4^{n-7}+$$

$$\frac{(n-5)(n-6)(n-7)(n-8)}{2\times3\times4}4^{n-9}-\ldots.]-2[4^{n-2}-(n-3)\,4^{n-4}+\frac{(n-4)(n-5)}{2}\,4^{n-6}$$

$$-\frac{(n-5)(n-6)(n-7)}{2\times3}4^{n-8}+\frac{(n-6)(n-7)(n-8)(n-9)}{2\times3\times4}4^{n-10}-\ldots.].$$

In finding values, we use those terms only which, in the substitutions for n, have *positive* and *zero* exponents.

Take $n=5$; then $x_5=4[4^4-3\times4^2+4^0]-2[4^3-2\times4']=724$.

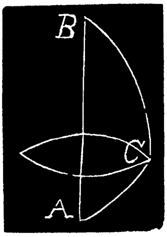
Excellent solutions were received from *G. B. M. ZERR*, *J. SCHEFFER*, and the late *J. H. DRUMMOND*.

AVERAGE AND PROBABILITY.

128. Proposed by **G. B. M. ZERR, A. M., Ph. D.**, Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

Two small circles are drawn on the surface of a sphere so as to intersect; find average area of the spherical triangle formed by joining the poles and one of the intersections of the small circles with arcs of great circles.

Solution by the PROPOSER.



Let A and B be the poles of the two small circles; C their intersection; join AC, BC, AB by arcs of great circles. Giving AB, AC, BC all the values within the limits of the problem, we are required to find the average area of ABC; and we may consider one of the poles, as A, to be fixed. Let r=radius of sphere, arc $AC=rx$, $BC=ry$, $AB=rz$, $\angle CAB=\theta$, $\angle CBA=\varphi$, $\angle ACB=\psi$, area $ABC=r^2(\theta+\phi+\phi-\pi)=u$.

Then $\cos z=\cos x\cos y+\sin x\sin y\cos\psi$....(1).

$\sin z\cos\theta=\sin x\cos y-\cos x\sin y\cos\psi$....(2).

$\sin z\cos\varphi=\cos x\sin y-\sin x\cos y\cos\psi$....(3).

An element of surface at B is $2\pi r^2\sin z dz$. The limits of x are 0 and $\frac{1}{2}\pi$; from $y=0$ to $y=x$, the limits of z are $x-y$ and $x+y$; and from $y=x$ to $y=\frac{1}{2}\pi$, the limits of z are $y-x$ and $y+x$. Hence the required average surface is

$$\Delta=\frac{\int_0^{\frac{1}{2}\pi}\left[\int_0^x\int_{x-y}^{x+y}urdy.2\pi r^2\sin zdz+\int_x^{\frac{1}{2}\pi}\int_{y-x}^{y+x}urdy.2\pi r^2\sin zdz\right]rdx}{\int_0^{\frac{1}{2}\pi}\left[\int_0^x\int_{x-y}^{x+y}rdy.2\pi r^2\sin zdz+\int_x^{\frac{1}{2}\pi}\int_{y-x}^{x+y}rdy.2\pi r^2\sin zdz\right]rdx}$$

$$=\frac{r^2}{2}\int_0^{\frac{1}{2}\pi}\left[\int_0^x\int_{x-y}^{x+y}(\theta+\varphi+\psi-\pi)dy\sin zdz+\int_x^{\frac{1}{2}\pi}\int_{y-x}^{x+y}(\varphi+\theta+\psi-\pi)dy\sin zdz\right]dx$$

$$\int(\theta+\varphi+\psi-\pi)\sin zdz=-2\theta\cos^2\tfrac{1}{2}z-2\varphi\cos^2\tfrac{1}{2}z-\psi\cos z+\pi\cos z+2\int\cos^2\tfrac{1}{2}zd\theta$$

$$+2\int\cos^2\tfrac{1}{2}zd\varphi+\int\cos zd\psi.$$

Now $d\theta=-\sin y\cos\varphi\operatorname{cosec}zd\psi$....(4), $d\varphi=-\sin x\cos\theta\operatorname{cosec}zd\psi$....(5).

From (1) and (3), $\sin^2\frac{1}{2}z=\sin^2\frac{1}{2}(x+y)-\sin x\sin y\cos^2\frac{1}{2}\psi$....(6).

$\sin z\cos\varphi=\sin(x+y)-2\sin x\cos y\cos^2\frac{1}{2}\psi$....(7).

From (4), (6), and (7),

$$2\int\cos^2\tfrac{1}{2}zd\theta=-\int\frac{2\sin y\cos^2\frac{1}{2}z\sin z\cos\varphi\,d\psi}{\sin^2z}=-\int\frac{\sin y\sin z\cos\varphi d\psi}{2\sin^2\frac{1}{2}z}$$

$$=-\tfrac{1}{2}\sin y\int\left(\frac{\sin(x+y)-2\sin x\cos y\cos^2\frac{1}{2}\psi}{\sin^2\frac{1}{2}(x+y)-\sin x\sin y\cos^2\frac{1}{2}\psi}\right)d\psi$$

$$=-\psi\cos y+2\tan^{-1}\left(\frac{\sin\frac{1}{2}(x+y)\tan\frac{1}{2}\psi}{\sin\frac{1}{2}(x-y)}\right).$$

Similarly, $2\int\cos^2\frac{1}{2}zd\varphi=-\psi\cos x+2\tan^{-1}\left(\frac{\sin\frac{1}{2}(y+x)\tan\frac{1}{2}\psi}{\sin\frac{1}{2}(y-x)}\right)$.

$$\int\cos zd\psi=\int(\cos x\cos y+\sin x\sin y\cos\psi)d\psi=\psi\cos x\cos y+\sin x\sin y\sin\psi.$$

When $z=x+y$, $\theta=\varphi=0$, $\psi=\pi$; when $z=x-y$, $\theta=\psi=0$, $\theta=\pi$; when $z=y-x$, $\varphi=\psi=0$, $\theta=\pi$.

$$\therefore\ \triangle=\tfrac{1}{2}\pi r^2\int_0^{\frac{1}{2}\pi}\left[\int_0^x(3+\cos x\cos y-\cos x-\cos y)dy\right.$$

$$\left.+\int_x^{\frac{1}{2}\pi}(3+\cos x\cos y-\cos x-\cos y)dy\right]dx.$$

$$\therefore\ \triangle=\tfrac{1}{2}\pi r^2\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}(3+\cos x\cos y-\cos x-\cos y)dxdy$$

$$=\tfrac{1}{4}\pi r^2\int_0^{\frac{1}{2}\pi}(3\pi+2\cos x-\pi\cos x-2)dx=\tfrac{1}{8}\pi r^2(3\pi^2+4-4\pi)$$

129. Proposed by J. K. ELLWOOD, Principal of Colfax School, Pittsburg, Pa.

A and B play with two dice, A throwing. If he throws 7 or 11, he wins; if he throws 3, or two aces, or two sixes, B wins. But if he throws 4, 5, 6, 8, 9, or 10, he continues throwing to duplicate this throw, in which event he wins; if in throwing, however, he throws 7, B wins. What is the expectancy of each? [This is the regulation "crap" game, B being banker.]

Solution by G. B. M. ZERR, A..M., Ph.D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

The chance of throwing 7 or 11 is $\frac{2}{9}$; the chance of throwing 2, 3, or 12 is $\frac{1}{9}$; the chance of throwing 4, 5, 6, 8, 9, or 10 is $\frac{2}{3}$. If A throws 4 the first throw the chance of winning the second throw is $\frac{1}{12}.\frac{2}{3}$; of winning the third throw is $\frac{1}{12}.\frac{2}{3},[1-(\frac{1}{12}+\frac{1}{6})]=\frac{1}{12}.\frac{2}{3}.\frac{3}{4}$.

$\therefore$ A's chance of winning on 4 is $\frac{2}{9}+\frac{1}{12}.\frac{2}{3}[1+\frac{3}{4}+(\frac{3}{4})^2+(\frac{3}{4})^3+....]=\frac{4}{9}$.

A's chance of winning on 5 is $\frac{2}{9}+\frac{1}{9}.\frac{2}{3}[1+\frac{13}{18}+(\frac{13}{18})^2+(\frac{13}{18})^3+.,..]=\frac{22}{45}$.

A's chance of winning on 6 is $\frac{2}{9}+\frac{5}{36}.\frac{2}{3}[1+\frac{25}{36}+(\frac{25}{36})^2+(\frac{25}{36})^3+....]=\frac{52}{99}$.

A's chance of winning on 8, 9, or 10 is the same as for 6, 5, or 4.

$\therefore$ A's chance$=\frac{1}{3}(\frac{4}{9}+\frac{22}{45}+\frac{52}{99})=\frac{722}{1485}$; B's chance$=1-\frac{722}{1485}=\frac{763}{1485}$.

If the wager is given their expectation follows at once.

Also solved, with different results, by *W. W. LANDIS*.

MISCELLANEOUS.

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128. Proposed by J. E. SANDERS, Hackney, O.

The sides of a trapezium are $a=29$, $b=32$, $c=40$, and $d=36$. If c is opposite a, and the diagonals equal, what is the length of either diagonal?